

# THE PUISEUX CHARACTERISTIC OF A SMALL GROWTH VECTOR

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**ABSTRACT.** Germs of Goursat distributions can be classified according to a geometric coding called an RVT code. Jean ([3]) and Mormul([7], [8]) have shown that this coding carries precisely the same data as the small growth vector. Montgomery and Zhitomirskii ([5]) have shown that such germs correspond to finite jets of Legendrian curve germs, and that the RVT coding corresponds to the classical invariant in the singularity theory of planar curves: the Puiseux characteristic. Here we derive a simple formula, Theorem 2.1, for the Puiseux characteristic of the curve corresponding to a Goursat germ with given small growth vector. The simplicity of our theorem (compared with the more complex algorithms previously known) suggests a deeper connection between singularity theory and the theory of nonholonomic distributions.

*Key words and phrases.* Goursat flag, small growth vector, nonholonomic distribution, Puiseux characteristic.

2010 *Mathematics Subject Classification.* 58A30, 58A17, 58K50, 53A55.

## 1. INTRODUCTION

**1.1. Goursat distributions and small growth vectors.** Let  $M$  be a smooth manifold of real dimension  $n \geq 4$ . Let  $D \subset TM$  be a smooth distribution (subbundle) of corank  $s$ . Let  $D^2 = [D, D] + D$ , called the *Lie square* of  $D$ . Iterate this squaring to obtain a chain

$$D_s \subseteq D_{s-1} \subseteq \cdots \subseteq D_i \subseteq D_{i-1} \subseteq \cdots$$

where  $D_s = D$  and  $D_{i-1} = D_i^2$  for  $i \leq s$ . Note that  $D_i$  may not, in general, have constant rank, and thus fail to be a distribution on  $M$ .  $D$  is called *Goursat* if  $\text{corank } D_i = i$ . In this case, one has a *Goursat flag*  $\mathcal{F}$ :

$$D_s \subset D_{s-1} \subset \cdots \subset D_1 \subset D_0 = TM$$

Note that when  $D$  is Goursat, each member  $D_i$  of the flag is itself a distribution, and a hyperplane in  $D_{i-1}$ .

Given a Goursat distribution  $D$ , one can alternatively form the sequence  $D^{(i)} = [D, D^{(i-1)}] + D^{(i-1)}$ , where  $D^{(0)} = D$  and  $i \geq 1$ . It is not hard to show that this sequence will also eventually terminate. That is, there exists an  $r$  such that  $D^{(r)} = TM$ . Thus, Goursat distributions are *completely nonholonomic*. The least such  $r$  is called the *degree of nonholonomy*. For each  $p \in M$ , we define the *small growth vector at  $p$*  to be the integer valued vector

$$sgv(p) = (\dim D^{(0)}(p), \dim D^{(1)}(p), \dots, \dim D^{(r)}(p) = n)$$

While the small growth vector is the traditional object of interest in the theory of nonholonomic distributions, for us it is more convenient to work with the *derived vector*, which is equivalent in our setting.

**Definition 1.1.** The *derived vector* of a small growth vector consists of the multiplicities of the entries in the small growth vector.

For a Goursat distribution, the dimensions of the sequence  $D^{(i)}$  grow by at most one at a time, so from the list of multiplicities we may recover the original small growth vector. By convention, we omit the last multiplicity 1 from the derived vector. For example, if we are given a small growth vector  $(2, 3, 4, 4, 5)$ , the associated derived vector is  $(1, 1, 2)$ . Similarly, given a derived vector  $(1, 1, 1, 3, 3)$ , the associated small growth vector is  $(2, 3, 4, 5, 5, 5, 6, 6, 6, 7)$ . It is not obvious, but follows from Jean's work, that the derived vector is always increasing.

**1.2. History.** Goursat distributions are the antithesis of integrable distributions, as they are *bracket-generating*. However, they grow the slowest of all such distributions. Cartan ([1]) studied the model of the canonical distribution on the jet space  $J^{n-2}(\mathbb{R}, \mathbb{R})$ . All Goursat distributions were believed to be equivalent to Cartan's until Giaro, Kumpera, and Ruiz discovered the first singularity in 1978 ([2]). A generic Goursat distribution is in fact equivalent to Cartan's (modulo trivial factors).

Montgomery and Zhitomirskii ([6]) introduced the Monster tower, a sequence of manifolds with distributions in which every Goursat germ occurs, along with a geometric coding in the letters RVT which is an invariant of the distribution germ with respect to diffeomorphic equivalence. Jean ([3]) studied the kinematic model of a car pulling  $N$  trailers and derived recurrence relations enabling one to compute the small growth vector of a distribution germ from the RVT code. Mormul ([7]) solved Jean's relations, allowing for the calculation of an RVT code from the small growth vector.

In [5], Montgomery and Zhitomirskii showed that Goursat germs correspond to finite jets of Legendrian curve germs, and that the RVT coding corresponds to the classical invariant in the singularity theory of planar curves: the Puiseux characteristic (see next section). They gave an explicit algorithm for computing the Puiseux characteristic from the RVT code.

**1.3. The Puiseux characteristic.** Suppose  $\gamma: (\mathbb{R}, 0) \rightarrow \mathbb{R}^2$  is the parametrization of an analytic plane curve germ. We say  $\gamma$  is *badly-parametrized* if there exist analytic germs  $\mu: (\mathbb{R}, 0) \rightarrow \mathbb{R}^2, \phi: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  such that  $d\phi/dt(0) = 0$  and  $\gamma = \mu \circ \phi$ . Otherwise,  $\gamma$  is called *well-parametrized*. If  $\gamma$  is well-parameterized and not immersed then we may define its Puiseux characteristic, an invariant with respect the RL-equivalence of curve germs. Then, up to RL-equivalence,  $\gamma$  has the form

$$\gamma(t) = (t^m, \sum_{k \geq m} a_k t^k)$$

where  $m \geq 2$ .

The definition of the Puiseux characteristic is the following. Let  $\lambda_0 = e_0 = m$ . Then define inductively for  $i \geq 0$

$$\lambda_{i+1} = \min\{k \mid a_k \neq 0, e_i \nmid k\}, \quad e_{i+1} = \gcd(e_i, \lambda_{i+1})$$

until we first obtain a  $g$  with  $e_g = 1$ . Then the vector  $[\lambda_0; \lambda_1, \dots, \lambda_g]$  is called the *Puiseux characteristic* of  $\gamma$ .

The Puiseux characteristic is the fundamental invariant in the singularity theory of plane curves. In [9], Proposition 4.3.8 shows that it is equivalent to at least seven other classical invariants.

## 2. MAIN RESULT

In the present contribution we effectively compose the algorithms presented in [7] and [5], yielding a formula for the Puiseux characteristic of the curve corresponding to a Goursat germ with given small growth vector. This formula turns out to be simpler than either of the two from which it was derived, suggesting a deeper geometric link between singularities of plane curves and singular Goursat distributions. The problem solved herein was first proposed in [5] as Question 9.19, part 3, and was asked again in [8] in the Afterword.

**2.1. Main Theorem.** Suppose we are given a small growth vector whose derived vector (see Definition 1.1) is

$$der = (d_1, \dots, d_N).$$

Consider the set  $S = \{d_i \mid d_{i-1} \text{ properly divides } d_i\}$ . Let  $g = |S|$ . For  $1 \leq j \leq g$  denote by  $d_{k_j}$  the entries of  $S$  in decreasing order. That is, write  $S = \{d_{k_1}, \dots, d_{k_g}\}$ , with  $d_{k_1} > \dots > d_{k_g}$ .

**Theorem 2.1.** The corresponding Puiseux characteristic is

$$[\lambda_0; \lambda_1, \dots, \lambda_g]$$

where

$$\begin{aligned} \lambda_0 &= d_N \\ \lambda_j &= \sum_{i \geq k_j} d_i + d_{k_j} + d_{k_j-1} \end{aligned}$$

for  $1 \leq j \leq g$ .

**2.2. Example.** Suppose  $der = (1, 1, 2, 2, 2, 2, 2, 2, 4, 6, 6, 6, 18, 24, 24)$ . Note that  $d_N = d_{15} = 24$ . We also have  $S = \{18, 4, 2\}$ , and therefore  $g = 3$ . Then write  $S = \{18, 4, 2\} = \{d_{13}, d_9, d_3\}$  so that  $k_1 = 13$ ,  $k_2 = 9$ , and  $k_3 = 3$ . Finally, we compute

$$\begin{aligned} \lambda_1 &= \sum_{i \geq 13} d_i + d_{13} + d_{12} = 90 \\ \lambda_2 &= \sum_{i \geq 9} d_i + d_9 + d_8 = 94 \\ \lambda_3 &= \sum_{i \geq 3} d_i + d_3 + d_2 = 103 \end{aligned}$$

The Puiseux characteristic is thus

$$[24; 90, 94, 103]$$

**2.3. Remark.** For the proof of the theorem, we need alternative notation which captures the multiplicities of repeated entries in the derived vector. To this end, we rewrite the derived vector as

$$der = (\underbrace{M_1, M_1, \dots, M_1}_{m_1}, \underbrace{M_2, M_2, \dots, M_2}_{m_2}, \dots, \underbrace{M_{v+1}, M_{v+1}, \dots, M_{v+1}}_{m_{v+1}}).$$

Assume that  $m_1 = M_2$ . This ensures that the vector represents a critical RVT code (that is, one which ends with the letter V or T). We can only discuss the Puiseux characteristic for critical (non-immersed) plane curves, since any immersed curve germ has normal form  $(t, 0)$ . Critical RVT codes correspond to critical curves. Then  $S = \{M_i \mid M_{i-1} \text{ divides } M_i\}$ . Let  $N_1, N_2, \dots, N_g$  denote the elements of  $S$  in decreasing order. We always have  $N_g = M_2$ , since  $M_1 = 1$ . For  $1 \leq j \leq g$  let  $M_{k_j} = N_j$ . In this notation, Theorem 2.1 says that the corresponding Puiseux characteristic is

$$[\lambda_0; \lambda_1, \dots, \lambda_g]$$

where

$$\lambda_0 = M_{v+1} \tag{1}$$

$$\lambda_j = \sum_{i \geq k_j} m_i M_i + M_{k_j} + M_{k_j-1} \tag{2}$$

for  $1 \leq j \leq g$ .

**2.4. Mormul's results.** Before proving the Theorem, we recall Mormul's construction of the RVT code from the derived vector (see [7]), with slightly modified notation.

We have that  $v$  is the number of letters V in the RVT code  $\alpha$ . We have thus (as in section 3.1) partitioned our code in  $v + 1$  pieces separated by the letters V. We write that the *last* letter V in the code is followed by  $t_1$  many letters T, and then  $r_1$  many letters R. We continue for  $1 \leq i \leq v$  letting  $t_i$  denote the number of letters T following the  $i$ th V *from the right*, and  $r_i$  denote the number of letters R following those letters T. Let  $r_{v+1}$  denote the number of letters R preceding the first letter V. Mormul derived the following relations:

$$r_{v+1} = m_{v+1} + 1$$

$$t_1 = M_2 - 2$$

$$r_1 = m_1 - M_2$$

For  $2 \leq j \leq v$  we have

**Case 1:**  $M_j$  divides  $M_{j+1}$ . Then

$$t_j = \frac{M_{j+1}}{M_j} - 2$$

$$r_j = m_j - t_j - 1$$

**Case 2:**  $M_j$  does not divide  $M_{j+1}$ . Then

$$t_j = m_j - 1$$

$$r_j = 0$$

We now prove the theorem.

### 3. PROOF OF THEOREM

**3.1. Proof of Theorem;  $v=1$ .** Assume  $v = 1$ , so we need only compute the multiplicities  $r_2, t_1$ , and  $r_1$ . Recall that  $m_1 = M_2$  here as well. From Mormul's

relations, we have

$$\begin{aligned} r_2 &= m_2 + 1 \\ t_1 &= M_2 - 2 \\ r_1 &= m_1 - t_1 - 2 = 0 \end{aligned}$$

Thus, the given derived vector corresponds to the RVT code  $(\alpha) = R^{m_2+1}VT^{M_2-2}$ . We now compute the Puiseux characteristic following Montgomery and Zhitomirskii (section 3.8.4 in [5]).

$$(a, b) = \mathbb{E}_{VT^{M_2-2}}(1, 2) = \mathbb{E}_{VT^{M_2-3}}(1, 3) = \cdots = \mathbb{E}_V(1, M_2) = (M_2, M_2 + 1)$$

Thus, the Puiseux characteristic is  $\text{Pc}(der) = [\lambda_0; \lambda_1]$  where

$$\begin{aligned} \lambda_0 &= a = M_2 \\ \lambda_1 &= (m_2 - 1)a + a + b \\ &= m_2 M_2 + (M_2 + 1) \\ &= (m_2 + 1)M_2 + M_1 \end{aligned}$$

This proves (1) and (2) for  $v = 1$ . ■

**3.2. Proof of Theorem; Case A.** This is the case when  $g = 1$ . That is, we assume  $M_i$  does not divide  $M_{i+1}$  for all  $i \geq 2$ . Mornu's relations give

$$\begin{aligned} r_{v+1} &= m_{v+1} + 1 \\ t_1 &= M_2 - 2 \\ r_1 &= m_1 - M_2 = 0 \end{aligned}$$

and for  $j = 2, \dots, v$  we are in "Case 2" so that

$$t_j = m_j - 1, \quad r_j = 0.$$

Thus our RVT code is

$$(\alpha) = R^{m_{v+1}+1}VT^{m_v-1}VT^{m_{v-1}-1} \cdots VT^{m_3-1}VT^{m_2-1}VT^{M_2-2}$$

We can now see that this case corresponding to "Case A" in [5]. That is, our RVT code is of the form

$$(\alpha) = R^l \omega$$

where  $\omega$  is entirely critical. Here we have  $l = m_{v+1} + 1$ . Our task is now to compute  $(a, b) = \mathbb{E}_\omega(1, 2)$ . We make the following 2 claims:

*Claim 1:*  $a = M_{v+1}$

*Claim 2:*  $b = \sum_{i=2}^v m_i M_i + M_2 + M_1$

It then follows from [5] that the Puiseux characteristic is  $[\lambda_0; \lambda_1]$ , where

$$\begin{aligned} \lambda_0 &= a = M_{v+1} \\ \lambda_1 &= (l - 2)a + a + b \\ &= (m_{v+1} - 1)M_{v+1} + M_{v+1} + \sum_{i=2}^v m_i M_i + M_2 + M_1 \\ &= \sum_{i=2}^{v+1} m_i M_i + M_2 + M_1 \end{aligned}$$

which is the precisely the content of (1) and (2) in this case.

It remains to prove the claims. We note that the value  $l = m_{v+1} + 1$  is irrelevant here, and we assume without loss of generality that  $l = 2$ . We will prove the claims by induction on  $v$ . The case  $v = 1$  is done above. Now suppose  $v = 2$ , so that

$$(\alpha) = RRV T^{m_2-1} V T^{M_2-2}.$$

We now compute  $(a, b)$ , following [5]:

$$\begin{aligned} (a, b) &= \mathbb{E}_\omega(1, 2) = \mathbb{E}_{VT^{m_2-1}VT^{M_2-2}}(1, 2) \\ &= \mathbb{E}_{VT^{m_2-1}VT^{M_2-3}}(1, 3) \\ &= \mathbb{E}_{VT^{m_2-1}VT^{M_2-4}}(1, 4) \\ &\vdots \\ &= \mathbb{E}_{VT^{m_2-1}V}(1, M_2) \\ &= \mathbb{E}_{VT^{m_2-1}}(M_2, M_2 + 1) \\ &= \mathbb{E}_{VT^{m_2-2}}(M_2, 2M_2 + 1) \\ &= \mathbb{E}_{VT^{m_2-3}}(M_2, 3M_2 + 1) \\ &\vdots \\ &= \mathbb{E}_V(M_2, m_2M_2 + 1) \\ &= (m_2M_2 + 1, m_2M_2 + M_2 + 1) \\ &= (m_2M_2 + 1, m_2M_2 + M_2 + M_1) \end{aligned}$$

This proves claim 2 for  $v = 2$ , and gives  $a = m_2M_2 + 1$ . We next compute  $der(\alpha)$  following Mornu:

$$\begin{aligned} (1, 1) &\xrightarrow{V} (1, 1, 2) \\ &\xrightarrow{T} (1, 1, 1, 3) \\ &\xrightarrow{T} (1, 1, 1, 4) \\ &\vdots \\ &\xrightarrow{T} (1^{m_2}, m_2) \\ &\xrightarrow{T} (1^{m_2+1}, m_2 + 1) \\ &\xrightarrow{V} (1, 1, 2^{m_2}, 2m_2 + 1) \\ &\xrightarrow{T} (1, 1, 1, 3^{m_2}, 3m_2 + 1) \\ &\xrightarrow{T} (1, 1, 1, 1, 4^{m_2}, 4m_2 + 1) \\ &\xrightarrow{T} (1^5, 5^{m_2}, 5m_2 + 1) \\ &\vdots \\ &\xrightarrow{T} (1^{M_2}, M_2^{m_2}, m_2M_2 + 1) \end{aligned}$$

So  $M_3 = m_2M_2 + 1 = a$ , proving Claim 1 for  $v = 1$ .

Both claims have been shown for  $v = 1$ , so we continue to the inductive step. Now let

$$\begin{aligned} der &= (M_1^{m_1}, M_2^{m_2}, \dots, M_v^{m_v}, M_{v+1}^{m_{v+1}}) \\ der' &= (M_1^{m_1}, M_2^{m_2}, \dots, M_v^{m_v}) \end{aligned}$$

with associated RVT codes

$$\begin{aligned} (\alpha) &= R^{m_{v+1}+1} VT^{m_v-1} VT^{m_{v-1}-1} \dots VT^{m_3-1} VT^{m_2-1} VT^{M_2-2} \\ (\alpha') &= R^{m_v+1} VT^{m_{v-1}-1} \dots VT^{m_3-1} VT^{m_2-1} VT^{M_2-2} \end{aligned}$$

so that

$$\begin{aligned} (\omega) &= VT^{m_v-1} VT^{m_{v-1}-1} \dots VT^{m_3-1} VT^{m_2-1} VT^{M_2-2} \\ (\omega') &= VT^{m_{v-1}-1} \dots VT^{m_3-1} VT^{m_2-1} VT^{M_2-2}. \end{aligned}$$

Then by induction, we have

$$(a', b') = \mathbb{E}_{\omega'}(1, 2) = (M_v, \sum_{i=2}^{v-1} m_i M_i + M_2 + M_1).$$

So we finally have

$$\begin{aligned} (a, b) &= \mathbb{E}_{\omega}(1, 2) = \mathbb{E}_{VT^{m_v-1}}(a', b') \\ &= \mathbb{E}_{VT^{m_v-1}}(M_v, b') \\ &= \mathbb{E}_{VT^{m_v-2}}(M_v, b' + M_v) \\ &\vdots \\ &= \mathbb{E}_V(M_v, b' + (m_v - 1)M_v) \\ &= (b' + (m_v - 1)M_v, b' + m_v M_v) \\ &= \left( \sum_{i=2}^v m_i M_i + M_2 + M_1 - M_v, \sum_{i=2}^v m_i M_i + M_2 + M_1 \right) \end{aligned}$$

This proves Claim 2, and reduces Claim 1 to showing

$$M_{v+1} = \sum_{i=2}^v m_i M_i + M_2 + M_1 - M_v \quad (3)$$

To show (3), we apply induction on  $v$ . That is, we assume

$$M_v = \sum_{i=2}^{v-1} m_i M_i + M_2 + M_1 - M_{v-1}.$$

Then

$$M_{v+1} + M_v = M_{v+1} + \sum_{i=2}^{v-1} m_i M_i + M_2 + M_1 - M_{v-1}$$

so that (3) reduces to

$$M_{v+1} - M_{v-1} = m_v M_v.$$

This last equality, however, follows from Proposition 3.2 and Theorem 3.3 of [7] (as does the base case  $v = 2$  in the above induction). This completes the proof of Case A. ■

**3.3. Proof of Theorem; Case B.** This is the case when  $g > 1$ . Mormul's results imply that this corresponds to multiple critical strings in the associated RVT code  $(\alpha)$ . The proof will be by induction on  $g$ , where the base case was completed in the previous section. The idea is to truncate the code  $(\alpha)$  after the last occurring letter R. To this end, we need only consider the entry  $N_{g-1} = M_{k_{g-1}}$  in *der*. For sake of cleaner notation, we set  $r = k_{g-1}$ . Then by assumption we have that  $M_{j-1}$  divides  $M_j$ , and  $M_{j-1}$  is the smallest such entry (besides  $M_1 = 1$ ). Then Mormul's relations imply that the RVT code has the form

$$(\alpha) = (\beta R^s \omega)$$

where

$$\omega = VT^{m_{r-1}-1} VT^{m_{r-2}-1} \dots VT^{m_2-1} VT^{M_2-2}$$

$$s = m_r - \frac{M_{r+1}}{M_r} + 1$$

$$\beta = \text{some critical RVT code}$$

In fact, we can say slightly more about  $(\beta)$  - it ends with  $VT^{M_{r+1}/M_r-2}$  - but we will not use this fact. We will adorn all data concerning  $(\beta)$  with a tilde to distinguish it from that of  $(\alpha)$ . In particular, we write  $[\tilde{\lambda}_0; \tilde{\lambda}_1, \dots, \tilde{\lambda}_{g-1}]$  for the Puiseux characteristic of  $(\beta)$ , and

$$(\tilde{M}_1^{\tilde{m}_1}, \tilde{M}_2^{\tilde{m}_2}, \dots, \tilde{M}_{\tilde{v}+1}^{\tilde{m}_{\tilde{v}+1}})$$

for the derived vector. Note that  $\tilde{g}$  is indeed equal to  $g - 1$  by construction. While not obvious from Mormul's relations listed above, in "Case 1" one always has  $r_j > 0$ . This essentially follows from a result of Luca and Risler (see [4]) which states that the degree of nonholonomy of a Goursat distribution on an  $n$ -manifold cannot exceed the  $n$ th Fibonacci number. (Here the degree of nonholonomy is equal to the sum of the entries in the derived vector plus one).

Now a result of Mormul (Proposition 1, [8]) allows us to compute the derived vector for  $(\beta)$  in terms of the derived vector for  $(\alpha)$ :

$$\begin{aligned} \tilde{m}_1 &= \frac{M_{r+1}}{M_r} \\ \tilde{v} &= v + r - 1 \\ \tilde{M}_i &= \frac{M_{r+i-1}}{M_r} \quad \text{for } i = 2, \dots, \tilde{v} + 1 \\ \tilde{m}_{i+1} &= m_{r+i} \quad \text{for } i = 1, \dots, v - r + 1 \end{aligned}$$

Now by induction we may assume

$$\begin{aligned} \tilde{\lambda}_0 &= \tilde{M}_{\tilde{v}+1} \\ \tilde{\lambda}_j &= \sum_{i \geq \tilde{k}_j} \tilde{m}_i \tilde{M}_i + \tilde{M}_{\tilde{k}_j} + \tilde{M}_{\tilde{k}_j-1} \end{aligned}$$



for  $1 \leq j \leq g-1$ . So the above relations imply

$$\tilde{\lambda}_0 = \frac{M_{v+1}}{M_r}.$$

Now the two claims from the previous section imply

$$(a, b) = \mathbb{E}_\omega(1, 2) = (M_r, \sum_{i=2}^{r-1} m_i M_i + M_2 + M_1).$$

Then the algorithm described in [5] gives

$$\lambda_0 = a\tilde{\lambda}_0 = M_r \frac{M_{v+1}}{M_r} = M_{v+1},$$

agreeing with (1) as desired.

We now compute  $\lambda_g$ . We always have  $M_1 = 1$  so  $N_g = M_2$  so  $k_g = 2$ . We first observe that

$$\begin{aligned} M_r \tilde{\lambda}_{g-1} &= \sum_{i \geq 2} m_{r+i-1} M_{r+i-1} + M_{r+1} + M_r \\ &= \sum_{i \geq r+1} m_i M_i + M_{r+1} + M_r \end{aligned}$$

Thus, following [5] we obtain

$$\begin{aligned} \lambda_g &= a(\tilde{\lambda}_{g-1} + s - 1) + b - a \\ &= M_r \left( \tilde{\lambda}_{g-1} + m_r - \frac{M_{r+1}}{M_r} \right) + \sum_{i=2}^{r-1} m_i M_i + M_2 + M_1 - M_r \\ &= M_r \tilde{\lambda}_{g-1} + M_r m_r + \sum_{i=2}^{r-1} m_i M_i + M_2 + M_1 - M_{r+1} - M_r \\ &= \sum_{i=2}^r m_i M_i + M_2 + M_1 + M_r \tilde{\lambda}_{g-1} - M_{r+1} - M_r \\ &= \sum_{i \geq 2} m_i M_i + M_2 + M_1 \end{aligned}$$

agreeing with (2) for  $j = g$ .

Finally, we compute the remaining entries in the Puiseux characteristic:  $\lambda_j$  for  $j = 1, \dots, g-1$ . First we recall that  $k_j$  is defined so that  $N_j = M_{k_j}$  is the  $j$ th smallest entry in  $\text{der}$  which is divisible by the preceding entry. Since this observation applies to the derived vectors of both  $(\alpha)$  and  $(\beta)$ , we find that

$$\begin{aligned} \tilde{M}_{\tilde{k}_j-1} \text{ divides } \tilde{M}_{\tilde{k}_j} &\Leftrightarrow \frac{M_{r+\tilde{k}_j-2}}{M_r} \text{ divides } \frac{M_{r+\tilde{k}_j-1}}{M_r} \\ &\Rightarrow M_{r+\tilde{k}_j-2} \text{ divides } M_{r+\tilde{k}_j-1} \\ &\Rightarrow M_{r+\tilde{k}_j-1} = M_{k_j} \\ &\Rightarrow r + \tilde{k}_j - 1 = k_j. \end{aligned}$$

Whence we obtain our result (2):

$$\begin{aligned}
\lambda_j &= a\tilde{\lambda}_j \\
&= M_r \sum_{i \geq \tilde{k}_j} \tilde{m}_i \tilde{M}_i + \tilde{M}_{\tilde{k}_j} + \tilde{M}_{\tilde{k}_j-1} \\
&= \sum_{i \geq \tilde{k}_j} m_{r+i-1} M_{r+i-1} + M_{r+\tilde{k}_j-1} + M_{r+\tilde{k}_j-2} \\
&= \sum_{i \geq k_j} m_i M_i + M_{k_j} + M_{k_j-1}
\end{aligned}$$

■

**Acknowledgments.** The author would like to thank Richard Montgomery, Piotr Mormul, Wyatt Howard and Alex Castro for helpful discussion, advice, and most importantly, motivation.

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